Multi-region relaxed Hall magnetohydrodynamics with flow

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The recent formulations of multi-region relaxed magnetohydrodynamics (MRxMHD) have generalized the famous Woltjer-Taylor states by incorporating partial relaxation and flow. In this paper, we generalize MRxMHD with flow to include Hall effects, and thereby obtain the partially relaxed counterparts of the famous double Beltrami states as a special subset. We demonstrate that our results (in the ideal MHD limit) constitute an important subset of ideal MHD equilibria, and we also compare our approach against other variational principles presented in the literature.

I. INTRODUCTION

Amongst all fluid descriptions of plasmas, none has proven to be as simple, relevant and versatile as ideal magnetohydrodynamics (MHD). Consequently, it has been widely employed in modelling fusion [1] and astrophysical [2] plasmas. One of the most crucial aspects of ideal MHD entails the study of its equilibria. A great deal of attention has been centred around the idea proposed by [3], in that the plasma energy can be extremized, subject to certain constraints, namely the so-called "ideal-constraints" that prevent topological variations of the magnetic field. In contrast, the Woltjer-Taylor state [4–6] is obtained by extremizing the magnetic energy, subject to holding the global magnetic helicity fixed (and constraints on the global flux and a boundary condition). This allows a wider class of equilibrium solutions to be accessed. We note that generalizations of the Woltjer-Taylor state to include flow have also been widely studied; see e.g. [7–10] for some early treatments of this subject.

The crucial assumption most commonly invoked in computing 3D equilibria in ideal MHD is the existence of continuously nested flux surfaces. It is possible to relax this assumption, insisting instead that only a finite number of flux surfaces are present, thereby constituting a case of *partial* relaxation. A model that gives rise to such equilibria can be seen as the generalization of the Taylor model, and originated in the studies undertaken by [11–13], and was dubbed multi-region relaxed MHD (MRxMHD). Subsequently, MRxMHD has been studied extensively, with a view towards understanding and extending it, in the works of [14–21]. The stepped pressure equilibrium code (SPEC) was presented in [22] based on MRxMHD, and it has been subsequently employed in multiple contexts such as reverse field pinches [23], magnetic islands and current sheets [24, 25], and pressure-driven amplification [26].

A common feature of ideal MHD and MRxMHD is that they rely on a similar set of constraints in arriving at the corresponding relaxed states. However, it is a well-known that fluid models more encompassing than ideal MHD are existent in the literature. Such fluid effects play an important role in certain regimes, especially in space and astrophysical plasmas [27]. It is, thus, natural to formulate relaxation theories for these models along the lines of Woltjer and Taylor. The most widely studied amongst them is Hall MHD [28–35], but equivalent treatments of two-fluid [36–38] and multi-fluid [39, 40] models can also be found.

Given the existence of two complementary approaches that generalize Taylor relaxation, viz. MRxMHD and Hall MHD, it is natural to look for a relaxation theory that encompasses both approaches. Indeed, this is the primary objective of this paper – to construct a MRxMHD theory with Hall effects, which we christen henceforth as multi-region, relaxed, Hall MHD, or MRxHMHD. For instance, owing to the property that Hall MHD is a singular perturbation of ideal MHD, we show that the partial relaxed states obtained from MRxHMHD are quite different from their MHD counterparts derived in [19, 20].

The outline of the paper is as follows. The relevant background material for carrying out the variational principle is presented in Section II. A detailed variation is carried out in Section III, leading to the final partial relaxed states of MRxHMHD. We compare the MRxMHD states with flow against ideal MHD equilibria, and offer a few general comments, in Section IV. Finally, we conclude with a summary of our results and prospects for future work in Section V.

II. MATHEMATICAL PRELIMINARIES AND THE VARIATIONAL PRINCIPLE

In this Section, we shall present some of the relevant mathematical properties of Hall MHD and then set up the procedure to obtain the relaxed states of MRxHMHD.

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A. The equations and properties of Hall MHD

The governing equations of Hall MHD are

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \mathbf{V}), \tag{1}$$

$$\frac{\partial \mathbf{V}}{\partial t} + \mathbf{V} \cdot \nabla \mathbf{V} = \rho^{-1} \nabla p + \rho^{-1} (\nabla \times \mathbf{B}) \times \mathbf{B}, \tag{2}$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{V} \times \mathbf{B}) - d_i \nabla \times \left(\rho^{-1} (\nabla \times \mathbf{B}) \times \mathbf{B}\right). \tag{3}$$

Note that the equations have been normalized in Alfvénic units, such that $d_i = \lambda_i / L$ is the normalized ion skin depth, where $\lambda_i = c/(\omega_{pi} L)$ is the ion skin depth in fiducial units and L is the appropriate scale length. Moreover, we note that p is the total pressure and it is assumed to be barotropic and adiabatic, i.e. it obeys the relation $p = \sigma \rho^\gamma$, where σ is a proportionality constant. ρ, $\mathbf{V}$ and $\mathbf{B}$ are the total mass density, the center-of-mass fluid velocity and the magnetic field respectively, and constitute the dynamical variables of interest.

It is worth remarking that Hall MHD has a (noncanonical) Hamiltonian formulation [41, 42]. The Hamiltonian formulation of Hall MHD is particularly useful in extracting a special class of invariants known as the Casimirs [42, 43] given by

$$M = \int_\Omega \rho \, d^3\tau, \tag{4}$$

$$K = \frac{1}{2} \int_\Omega \mathbf{B} \cdot \mathbf{A} \, d^3\tau, \tag{5}$$

$$C = \frac{1}{2} \int_\Omega \mathbf{P} \cdot (\nabla \times \mathbf{P}) \, d^3\tau, \tag{6}$$

where $\mathbf{P} = \mathbf{A} + d_i \mathbf{V}$ and is, sometimes, referred to as the (ion) canonical momentum. Similarly, we note that (6) is often referred to as the canonical helicity or the generalized helicity [28, 29].

A few other points, which we shall call upon later, must also be stated:

1. If we introduce the electron velocity $\mathbf{V}_e = \mathbf{V} - d_i \nabla \times \mathbf{B}/\rho$, it is easy to show that (3) becomes akin to the ideal MHD induction equation except for $\mathbf{V} \rightarrow \mathbf{V}_e$. As a result, it is still viable to speak of the magnetic flux being conserved, but it is advected along the electron 'particle' trajectory (in the Lagrangian picture) [44].

2. If we replace $\mathbf{V}$ with $\mathbf{V}_e$ in (1), it is easy to verify that the expression remains unchanged. This follows from the vector identity that the divergence of a curl vanishes. As a result, one may envision even the density being advected along the electron trajectory.

3. In [29], it was pointed out that Hall MHD (in the barotropic or incompressible limit) could be cast into a pair of coupled vorticity-type equations. Based on this result, it is possible to construct a canonical vorticity flux $\int \nabla \times \mathbf{P} \cdot d\mathbf{S}$, which is advected along the *ion* trajectory in the Lagrangian formalism. [45].

4. Thus, it is viable to consider all variables being advected along the electron trajectory, except for the composite variable $\mathbf{P}$ that is advected along the ion trajectory.

B. The MRxHMHD variational principle

We shall consider a plasma system that consists of N nested plasma regions R_l, separated by Hall MHD barriers. The energy is

$$E = \sum_l E_l = \int_{R_l} \left\{ \rho \frac{V^2}{2} + \frac{\sigma_l}{\gamma - 1} \rho^\gamma + \frac{B^2}{2} \right\} d^3\tau, \tag{7}$$

whilst the mass, magnetic helicity and canonical helicity carry over from (4), (5) and (6) respectively. In the multiregion picture, they correspond to

$$M_l = \int_{R_l} \rho \, d^3\tau, \tag{8}$$

$$K_l = \frac{1}{2} \int_{R_l} \mathbf{A} \cdot \mathbf{B} \, d^3\tau, \tag{9}$$

$$\begin{aligned} C_l &= \frac{1}{2} \int_{R_l} (\mathbf{A} + d_i \mathbf{V}) \cdot (\mathbf{B} + d_i \nabla \times \mathbf{V}) \, d^3\tau \\ &\equiv \frac{1}{2} \int_{R_l} \mathbf{P} \cdot (\nabla \times \mathbf{P}) \, d^3\tau, \end{aligned} \tag{10}$$

where the second equality follows from the relation $\mathbf{P} = \mathbf{A} + d_i \mathbf{V}$. In addition to the above constraints, we also consider the toroidal component of the angular momentum, following the approach of [19, 20], as an additional constraint. Thus, we have

$$L_l = \widehat{\mathbf{z}} \cdot \int_{R_l} \rho \mathbf{r} \times \mathbf{V} \, d^3\tau = \int_{R_l} \rho R \, \mathbf{V} \cdot \widehat{\phi} \, d^3\tau, \tag{11}$$

where R denotes the cylindrical radius, given that we are operating in this coordinate system. There are subtleties associated with toroidal angular momentum conservation, and we refer the reader to [19, 20] for a detailed discussion of the same. In essence, [19] state that an axisymmetric boundary (interface) that preserves this property during the relaxation process will ensure the conservation of (11). As this is undoubtedly a strong constraint, one can drop it if necessary, but the basic thrust of the analysis is not affected. It is worth pointing out that 3D MHD equilibria such as "snakes" [46] cannot be modelled, if we operate under the assumption that (11) is conserved.

We also need to specify the boundary conditions for our system. These conditions arise from the flux constraints, viz. the conservation of the magnetic flux and the canonical vorticity flux (see Point 3 of Sec. II A). The relevant details, in the case of ideal MHD, can be found in Sec. IV of [47]. When it comes to Hall MHD, the boundary conditions correspond to $\mathbf{B}\cdot\mathbf{n} = 0$ and $\nabla\times\mathbf{P}\cdot\mathbf{n} = 0$; subtracting the former from the latter leads to $\nabla\times\mathbf{V}\cdot\mathbf{n} = 0$ [33]. Following the approach outlined in [47], the flux constraints of Hall MHD translate to the equivalent conditions

$$\left(\mathbf{n}\times\delta\mathbf{A}\right) = -\left(\mathbf{n}\cdot\xi_e\right)\mathbf{B}, \tag{12}$$

$$\left(\mathbf{n}\times\delta\mathbf{P}\right) = -\left(\mathbf{n}\cdot\xi_i\right)\nabla\times\mathbf{P}. \tag{13}$$

In the above expressions, note that ξ_i and ξ_e stand for the ion and electron displacements respectively. We note that (13) involves ξ_i as the variable $\mathbf{P}$ exhibits ion advection, as noted in Section II A. Moreover, we also wish to point out the fact that (12) and (13) are *not* arbitrary. They are consequences of the frozen-in flux constraints, except that the former and latter are advected along different fluid trajectories.

The energy functional of the MRxHMHD reads as

$$W = \sum_l \Big\{ E_l - \nu_l\left(M_l - M_l^0\right) - \frac{1}{2}\mu_l\left(K_l - K_l^0\right) - \lambda_l\left(C_l - C_l^0\right) - \Omega_l\left(L_l - L_l^0\right)\Big\}, \tag{14}$$

where ν_l, μ_l, λ_l and Ω_l are the Lagrange multipliers. Also note that the X_l^0's are the constrained values of the respective X_l's.

A comment regarding the magnetic helicity (9) is due. In its present form, namely (9), the expression for the magnetic helicity is not gauge invariant. To ensure gauge invariance, one must include two loop integrals that encapsulate the amount of toroidal/poloidal flux contained within each region, and the loop integrals are computed about the inner (outer) boundary of a given region in the poloidal (toroidal) direction. For more details, the reader is referred to [18, 19]. However, it turns out that these new terms, even upon inclusion, do not contribute to the final result, and we have omitted them in our analysis for the sake of clarity. A similar line of reasoning is also valid when dealing with the canonical helicity (10). In both these respects, we follow the approach employed by [31] in their formulation and analysis of variational principles for two-fluid plasmas.

III. DERIVATION OF THE PARTIAL RELAXED STATES WITH HALL EFFECTS

In this Section, we shall use (14) as the functional subject to extremization, viz. we compute $\delta W = 0$. By doing so, we expect to compute the minimum energy states, but it must be recognized that a rigorous analysis will also necessitate taking the second variation of W to ensure that the final result is, indeed, a minima [48–50]. We shall leave such a procedure for future investigations, as most formulations of MRxMHD have not directly addressed this issue [18–20].

We also note that our approach belongs to the category of variational principles that extremize the energy. An alternative approach, also widely studied in the literature, is to extremize the entropy instead [10, 38] and a detailed discussion of this subject can be found in [14].

Before proceeding further, we begin by noting a useful identity [19, 47, 51] that we shall use in the subsequent derivations. It corresponds to

$$\delta\int_D X d^3\tau = \int_D \delta X d^3\tau + \int_{\partial D}\left(\mathbf{n}\cdot\xi\right) X d^2\sigma, \tag{15}$$

where D and ∂D are the volume and bounding surface respectively, X is the functional under consideration, and ξ corresponds to the fluid displacement.

A. The Energy functional

The first variation of the energy functional E_l can be obtained as follows

$$\delta E_l = \delta\int_{R_l}\rho\frac{V^2}{2}d^3\tau + \delta\int_{R_l}\frac{\sigma_l}{\gamma-1}\rho^\gamma d^3\tau + \delta\int_{R_l}\frac{B^2}{2}d^3\tau, \tag{16}$$

and the individual components yield

$$\delta\int_{R_l}\rho\frac{V^2}{2}d^3\tau = \int_{R_l}\delta\rho\frac{V^2}{2}d^3\tau + \int_{R_l}\delta\mathbf{V}\cdot\rho\mathbf{V}\,d^3\tau + \int_{\partial R_l}\left(\mathbf{n}\cdot\xi_e\right)\rho\frac{V^2}{2}d^2\sigma, \tag{17}$$

$$\delta\int_{R_l}\frac{\sigma_l}{\gamma-1}\rho^\gamma d^3\tau = \int_{R_l}\delta\rho\frac{\gamma}{\gamma-1}\sigma_l\rho^{\gamma-1}\,d^3\tau + \int_{\partial R_l}\left(\mathbf{n}\cdot\xi_e\right)\;\frac{\sigma_l}{\gamma-1}\rho^\gamma\,d^2\sigma, \tag{18}$$

$$\delta\int_{R_l}\frac{B^2}{2}d^3\tau = \int_{R_l}\delta\left(\frac{B^2}{2}\right)d^3\tau + \int_{\partial R_l}\left(\mathbf{n}\cdot\xi_e\right)\frac{B^2}{2}\,d^2\sigma, \tag{19}$$

The first term in (19) can be further simplified as follows

$$\int_{R_l}\delta\left(\frac{B^2}{2}\right)d^3\tau = \int_{R_l}\left(\nabla\times\delta\mathbf{A}\right)\cdot\left(\nabla\times\mathbf{A}\right)d^3\tau, \tag{20}$$

and we use the vector calculus identities

$$\nabla\cdot\left(\mathbf{F}\times\mathbf{G}\right) = \mathbf{G}\cdot\left(\nabla\times\mathbf{F}\right) - \mathbf{F}\cdot\left(\nabla\times\mathbf{G}\right), \tag{21}$$

$$\int_v \nabla\cdot\mathbf{F}\, d^3\tau = \int_{\partial v} \mathbf{n}\cdot\mathbf{F}\, d^2\sigma, \quad (22)$$

in (20) to obtain the final relation

$$\int_{R_l} \delta\left(\frac{B^2}{2}\right) d^3\tau = \int_{R_l} \delta\mathbf{A}\cdot(\nabla\times\mathbf{B})\, d^3\tau + \int_{\partial R_l} (\mathbf{n}\times\delta\mathbf{A})\cdot\mathbf{B}\, d^2\sigma. \quad (23)$$

We are now free to substitute (12) into the second term of (23). Next, we take the ensuing result and substitute it into (19). We obtain

$$\delta\int_{R_l} \frac{B^2}{2} d^3\tau = \int_{R_l} \delta\mathbf{A}\cdot(\nabla\times\mathbf{B})\, d^3\tau - \int_{\partial R_l} (\mathbf{n}\cdot\xi_e)\frac{B^2}{2}\, d^2\sigma, \quad (24)$$

and it is important to recognize that the second term on the RHS of (19) and (24) are the same, but *opposite* in sign.

B. The Helicity Functionals

Let us first begin with the magnetic helicity.

$$\delta K_l = \frac{1}{2}\int_{R_l} \delta(\mathbf{A}\cdot\mathbf{B})\ d^3\tau + \frac{1}{2}\int_{\partial R_l} (\mathbf{n}\cdot\xi_e)\,\mathbf{B}\cdot\mathbf{A}\ d^2\sigma, \quad (25)$$

and we can simplify the first term on the RHS by invoking $\mathbf{B} = \nabla\times\mathbf{A}$ along with (21) and (22) and carrying out a procedure akin to that undertaken for the magnetic energy. Upon simplification, we end up with

$$\delta K_l = \int_{R_l} \delta\mathbf{A}\cdot\mathbf{B}\, d^3\tau + \frac{1}{2}\int_{\partial R_l} \mathbf{A}\cdot[\mathbf{n}\times\delta\mathbf{A} + (\mathbf{n}\cdot\xi_e)\,\mathbf{B}]\ d^2\sigma, \quad (26)$$

and the second term on the LHS vanishes upon using the boundary condition (12). Thus, we have

$$\delta K_l = \int_{R_l} \delta\mathbf{A}\cdot\mathbf{B}\ d^3\tau. \quad (27)$$

Dealing with the canonical helicity is much harder owing to its greater complexity. However, we can bypass a great deal of this complexity if we use the canonical momentum $\mathbf{P}$ as our composite variable. Furthermore, note that (13) dictates that the boundary conditions are most naturally interpreted in terms of $\mathbf{P}$ as well, since this condition arises from the conservation of the canonical vorticity flux. From (II A), we recall that $\mathbf{P}$ is advected along the ion trajectory, and that it is written in terms of $\mathbf{P}$ as per the second line of (10). Using these facts, we find that

$$\delta C_l = \frac{1}{2}\int_{R_l} \delta(\mathbf{P}\cdot(\nabla\times\mathbf{P}))\ d^3\tau + \frac{1}{2}\int_{\partial R_l} (\mathbf{n}\cdot\xi_i)\,\mathbf{P}\cdot(\nabla\times\mathbf{P})\ d^2\sigma. \quad (28)$$

The second term on the RHS exhibits the label 'i' since it's advected along the ion trajectory. Although $\mathbf{P}$ is comprised of two dynamical variables, we note that the two vector calculus identities (21) and (22) are still valid. Hence, we apply them to the first term of (28) and end up with

$$\delta C_l = \int_{R_l} \delta\mathbf{P}\cdot(\nabla\times\mathbf{P})\ d^3\tau + \frac{1}{2}\int_{\partial R_l} \mathbf{P}\cdot[\mathbf{n}\times\delta\mathbf{P} + (\mathbf{n}\cdot\xi_i)\,\nabla\times\mathbf{P}]\ d^2\sigma, \quad (29)$$

and by using (13) in the second term on the RHS of the above expression, we see that it vanishes identically. Hence, we end up with

$$\delta C_l = \int_{R_l} \Big[\delta\mathbf{A}\cdot(\mathbf{B} + d_i\nabla\times\mathbf{V}) + d_i\,\delta\mathbf{V}\cdot(\mathbf{B} + d_i\nabla\times\mathbf{V})\Big]\ d^3\tau, \quad (30)$$

where we have used the fact that $\mathbf{P} = \mathbf{A} + d_i\mathbf{V}$ to rewrite our answer in terms of the dynamical variables.

C. The mass and angular momentum functionals

The variation of the angular momentum yields

$$\delta L_l = \int_{R_l} \delta\rho R\ \mathbf{V}\cdot\widehat{\phi}\ d^3\tau + \int_{R_l} \delta\mathbf{V}\cdot\rho R\widehat{\phi}\ d^3\tau + \int_{\partial R_l} (\mathbf{n}\cdot\xi_e)\left(\rho R\ \mathbf{V}\cdot\widehat{\phi}\right)\ d^2\sigma. \quad (31)$$

The variation of the mass functional leads us to the result

$$\delta M_l = \int_{R_l} \delta\rho\ d^3\tau + \int_{\partial R_l} (\mathbf{n}\cdot\xi_e)\,\rho\ d^2\sigma, \quad (32)$$

D. The derivation of the relaxed states and jump condition

Our final expressions are given by (17), (18), (24), (27), (30), (31) and (32). Upon setting $\delta W = 0$, we obtain

$$\nabla\times\mathbf{B} = (\mu_l + \lambda_l)\,\mathbf{B} + d_i\lambda_l\nabla\times\mathbf{V}, \quad (33)$$

$$\rho\mathbf{V} = d_i\lambda_l\mathbf{B} + d_i^2\lambda_l\nabla\times\mathbf{V} + \rho\Omega_l R\widehat{\phi}, \quad (34)$$

$$\nu_l = \frac{1}{2}V^2 + \frac{\gamma}{\gamma-1}\sigma_l\rho^{\gamma-1} - \Omega_l R\mathbf{V}\cdot\widehat{\phi}, \quad (35)$$

which have arisen from the variations with respect to $\mathbf{A}$, $\mathbf{V}$ and ρ respectively. In addition, we also end up with a bevy of surface integral terms collectively given by

$$\int_{R_l} (\mathbf{n}\cdot\xi_e)\left\{\rho\frac{V^2}{2} + \frac{\sigma_l}{\gamma-1}\rho^\gamma - \frac{B^2}{2} - \nu_l\rho - \Omega_l\left(\rho R\ \mathbf{V}\cdot\widehat{\phi}\right)\right\}\ d^2\sigma = 0, \quad (36)$$

and we are free to substitute (35) into the above expression. A lot of terms cancel leaving us with

$$\int_{R_l} (\mathbf{n}\cdot\xi_e)\left\{-\sigma_l\rho^\gamma-\frac{B^2}{2}\right\} d^2\sigma = 0, \tag{37}$$

and using the fact that $p_l=\sigma_l\rho^\gamma$, we arrive at the interface condition

$$\left[\left[p_l+\frac{B^2}{2}\right]\right]=0. \tag{38}$$

At this stage, it is worth comparing our results with the ideal MHD counterparts that were obtained in [19, 20]. Before proceeding further, we observe that the chief difference is that the canonical helicity of Hall MHD must be replaced by the cross helicity when dealing with ideal MHD. The rest of the invariants stay the same, implying that any differences that arise must be because of the cross vs canonical helicity difference.

Firstly, we commence by noting that the interface condition (38) is the same in both instances. This is not surprising since the interface condition is a manifestation of the force balance [52], and it is well known that the momentum equation of ideal and Hall MHD are identical to one another.

We also find that (35) is exactly identical to the ideal MHD result. This is not surprising since the helicities do not contribute to this equation, and therefore we do not expect any divergences in the result. Next, let us consider (33) and (34) and allow $d_i\to 0$. Upon comparing with [19, 20], we lose too many terms and the results do not match. At first glimpse, it may appear as though there was an error committed, but it is important to appreciate the mathematical fact that Hall MHD is a *singular perturbation* of ideal MHD [53] and the cross helicity does *not* follow from the canonical helicity simply by letting $d_i\to 0$, as indicated in [42, 54].

Instead, let us take a closer look at (34). We find that the second term on the RHS is much smaller than the first since it's $\mathcal{O}\left(d_i^2\right)$, whilst the first term is $\mathcal{O}\left(d_i\right)$. Hence, let us suppose that we drop *only* the $\mathcal{O}\left(d_i^2\right)$ in (33) and (34). Then, we find that our equations do indeed take on the same form (but with different coefficients) as the ideal MHD results from [19, 20].

Lastly, we point out an interesting subcase of our primary results. If we had started with an incompressible model, we would not have recovered (35). Moreover, if our system did not conserve toroidal angular momentum (which can arise in certain circumstances, as discussed in [19]), it amounts to setting $\Omega_l\to 0$ in (34) and (35). Under these conditions, it is easy to verify that the resulting set of equations are the multi-region equivalent of the famous double Beltrami states obtained for incompressible Hall MHD in [29]. In contrast, if the Hall effects were neglected (the ideal MHD limit), the above set of assumptions lead to a Woltjer-Taylor (single Beltrami) state along with the condition $\mathbf{V}\parallel\mathbf{B}$.

IV. A NOTE ON THE MRxMHD EQUILIBRIA

At this stage, we shall take a brief detour, and consider the MRxMHD equilibria derived in [19, 20]. As described above, a *careful* treatment of the partial relaxed states of MRxHMHD under the limit $d_i\to 0$ leads to the MRxMHD equilibria derived in [19]. We choose to focus on MRxMHD (instead of MRxHMHD) as we are interested in understanding how the equilibria of [19], which give rise to partially relaxed states with flow, compare against ideal MHD equilibria. We shall also contrast these states against alternative approaches presented in the literature.

A. Ideal MHD equilibria with flow

Let us begin by writing down the expressions for ideal MHD equilibria endowed with flow.

$$\nabla\cdot(\rho\mathbf{V})=0, \tag{39}$$

$$\rho\mathbf{V}\cdot\nabla\mathbf{V}=\mathbf{J}\times\mathbf{B}-\nabla p, \tag{40}$$

$$\nabla\times(\mathbf{V}\times\mathbf{B})=0, \tag{41}$$

where $p=\sigma\rho^\gamma$. It is easy to show that (40) can be rewritten as follows:

$$\boldsymbol{\omega}\times\mathbf{V}=\frac{\mathbf{J}\times\mathbf{B}}{\rho}-\nabla\left(\frac{\sigma\gamma\,\rho^{\gamma-1}}{\gamma-1}+\frac{V^2}{2}\right), \tag{42}$$

where we have introduced the notation $\boldsymbol{\omega}=\nabla\times\mathbf{V}$.

B. Partial relaxed states with flow

Here, we list the partially relaxed states with flow that were obtained in [19]. As mentioned earlier, we can recover these states by taking the limit $d_i\to 0$ in our model, although there are some subtleties involved. The relevant equations are

$$\nabla\times\mathbf{B}=\mu_l\mathbf{B}+\lambda_l\boldsymbol{\omega}, \tag{43}$$

$$\rho\mathbf{v}=\lambda_l\mathbf{B}, \tag{44}$$

$$\sigma_l\frac{\gamma\rho^{\gamma-1}}{\gamma-1}+\frac{V^2}{2}=\nu_l, \tag{45}$$

and we note that the label 'l' is present in the above equations, as we are looking at MRxMHD. In the continuum limit, this label can be dropped, and we shall do henceforth for the sake of simplicity.

C. Comparison of the two sets of equilibria

We shall compare the results of Sec. IV B against those of Sec. IV A.

We begin by observing that (44) can be expressed as $\mathbf{V} \parallel \mathbf{B}$, or $\mathbf{V} \times \mathbf{B} = 0$. When this condition is satisfied, it is easy to verify that (41) is automatically satisfied. Similarly, if we take the divergence of (44), we end up with $\nabla \cdot (\rho \mathbf{V}) = 0$ on account of $\nabla \cdot \mathbf{B} = 0$. This condition is exactly identical to (39). We turn our attention to (42) now, and (45) ensures that the second term on the RHS of (42) vanishes, i.e. the term inside the brackets. The remainder of (42) is given by

$$\boldsymbol{\omega} \times \mathbf{V} = \frac{\mathbf{J} \times \mathbf{B}}{\rho}, \tag{46}$$

and we shall show that (43) and (44) lead to the above relation. Let us take the cross product of $\mathbf{V}$ with (43). This leads us to

$$\mathbf{J} \times \mathbf{V} = \mathbf{B} \times \mathbf{V} + \lambda \boldsymbol{\omega} \times \mathbf{V}, \tag{47}$$

and we invoke the expression for $\mathbf{V}$, in terms of $\mathbf{B}$, which is given by (44). We substitute this expression into the LHS and the first term on the RHS of (47). This leads us to

$$\lambda \frac{\mathbf{J} \times \mathbf{B}}{\rho} = \lambda \boldsymbol{\omega} \times \mathbf{V}, \tag{48}$$

which is clearly identical to (46).

Thus, the purpose of this exercise is now complete. We have shown that the equilibria derived by [19] form a *valid subset* of ideal MHD equilibria. For this reason, it is plausible that the partial relaxed states derived in [19] (as well as the generalized states presented herein) constitute a physically meaningful set of MRxMHD equilibria with flow.

A few general observations regarding these partial relaxed states are in order. By substituting (44) into (43), we find that

$$\nabla \times \mathbf{B} = \mu \mathbf{B} + \lambda^2 \nabla \times \left(\frac{\mathbf{B}}{\rho}\right), \tag{49}$$

which is clearly a deformation of the Taylor state since $\mathbf{J} \times \mathbf{B} \neq 0$. In fact, we find that a near-Taylor state is recovered only in two limits that are outlined below.

- When $|\mathbf{V}| \ll |\mathbf{B}|$, we can drop the last term on the RHS of (43). This leads to a Taylor state to leading order.
- When the system is nearly incompressible, this ensures that $\rho \to$ const in (49), which in turn leads to $\mathbf{J} \times \mathbf{B} \to 0$.

The variational principles constructed herein, and in [19], were Eulerian in nature. A different variational formulation was presented in [21] that relied upon the use of Lagrangian variables and induced variations. Although the same interface condition, namely (38), was recovered, there were some differences in the two approaches. The final expressions in [21] corresponded to the Taylor state and the Euler equation for an ideal (neutral) fluid. It is straightforward to show that these equations also represent a valid set of the ideal MHD equilibria discussed in Sec. IV A. However, the relations obtained in [21] do not match the ones derived in [19], since the latter does not lead to a Taylor state, except under certain conditions.

The differences probably stem from the fact that ρ, $\mathbf{V}$ and $\mathbf{B}$ are treated as independent variables in the Eulerian picture. This is in sharp contrast to the Lagrangian treatment presented in [21], where the variations in ρ and $\mathbf{V}$ are expressed in terms of the displacement (along the lines of the methodology outlined in [55]). On the other hand, $\mathbf{B}$ and p are independent, and their variations are considered separately; see Eq. (3.21) of [21] for a discussion of the same.

The presence of induced variations also eliminated the need for the cross helicity (or, in our case, the canonical helicity) to be included in the variational principle. We note that this is quite different from most standard treatments in the literature, see e.g. [7, 9, 31, 36, 38]. It is likely that a clearer picture will emerge once the SPEC code [22] has been modified to implement flow. It will then be possible to compare the two approaches against experiments, or simulations from other sources, and thereby deduce their relative merits.

V. DISCUSSION AND CONCLUSION

The Woltjer-Taylor states of ideal MHD have proven to be widely successful in a host of fusion, space and astrophysical plasma environments. However, the implicit assumption of continous (and infinite) nested flux surfaces invoked in deriving such states can be relaxed. The resulting formulation, multi-region relaxed magnetohydrodynamics (MRxMHD), has proven to be successful in many contexts as noted in the Introduction.

Despite the great utility of MRxMHD, especially upon the inclusion of flow, it is still reliant on a variational principle that assumes the invariance of the magnetic and cross helicities, which are ideal MHD invariants. In this study, we have generalized MRxMHD further by adopting the framework of Hall MHD and invoking the magnetic helicity and the *canonical* helicity as the invariants in constructing our variational principle. The presence of the Hall term introduces some mathematical subtleties, given that Hall MHD retains residual two-fluid effects: one of them is manifest in the fact that the canonical vorticity $\nabla \times \mathbf{P} = \mathbf{B} + d_i \nabla \times \mathbf{V}$ is advected along the *ion* trajectory, whilst the magnetic field is advected along the *electron* trajectory, as pointed out in [44, 45, 56].

After going through the requisite algebra, we arrive at the final results, viz. the partial relaxed states given by (33), (34) and (35), and the interface condition (38). If we consider the incompressible limit of the former trio of equations, and assume that $\Omega_l \to 0$, the generalizations of the famous double Beltrami states [29] are duly obtained. Thus, MRxMHD with Hall effects (MRxHMHD) plays an analogous role to MRxMHD since the former leads to states akin to the double Beltrami states whilst the latter yields the Woltjer-Taylor (single Beltrami) states. As the double Beltrami states have been applied fairly successfully in both fusion and astrophysics, it is natural to suppose that the MRxHMHD equilibria will also prove to be useful in modelling the same phenomena.

We have also analyzed the MRxMHD equilibria obtained in [19], which form a subset of the equilibria derived in this paper. We showed that the partial relaxed states with flow that emerge from the Eulerian variational principle are a valid and meaningful subset of ideal MHD equilibria - a fact that lends further credence to our variational principle, and that of [19]. We also compared these results against the alternative approach espoused in [21], which gave rise to a different set of results, and indicated the potential factors that may be responsible for this outcome.

In subsequent studies, we hope to pursue two different lines of approach. We intend to employ the partial relaxed states derived in this paper to study systems where Hall effects play a role; one such example is to extend the approach presented in [57] to study the magnetospheres of the Jovian planets. Secondly, we are in the process of improving the successful SPEC code [22] to include flow, which can then be used to study a wide range of issues in fusion plasmas.

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